LECTURE 4: MARKOV CHAINS

Modeling and Simulation 2

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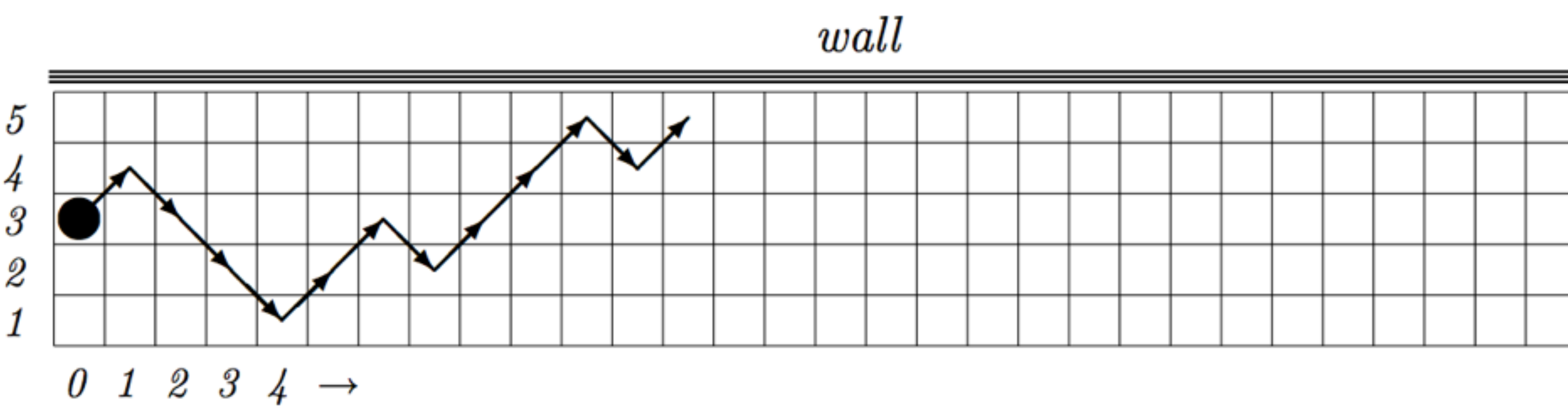
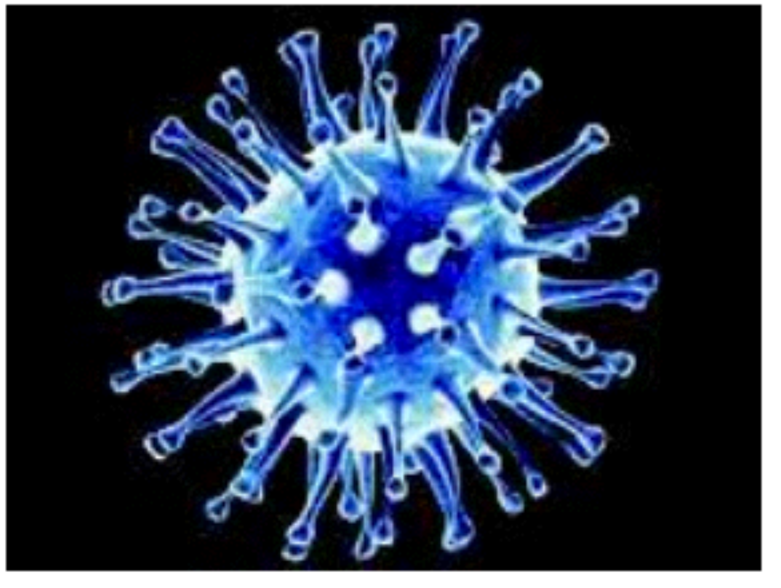
Winter 2014

Example 1: (Random walks)
A drunk walks along a pavement of width 5. At each time step he/she moves one position forward, and one position either to the left or to the right with equal probabilities. Except: when in position 5 can only go to 4 (wall), when in position 1 and going to the right the process ends (drunk falls off the pavement). How far will the walker get on average? What is the probability for the walker to arrive home when his/her home is K positions away?



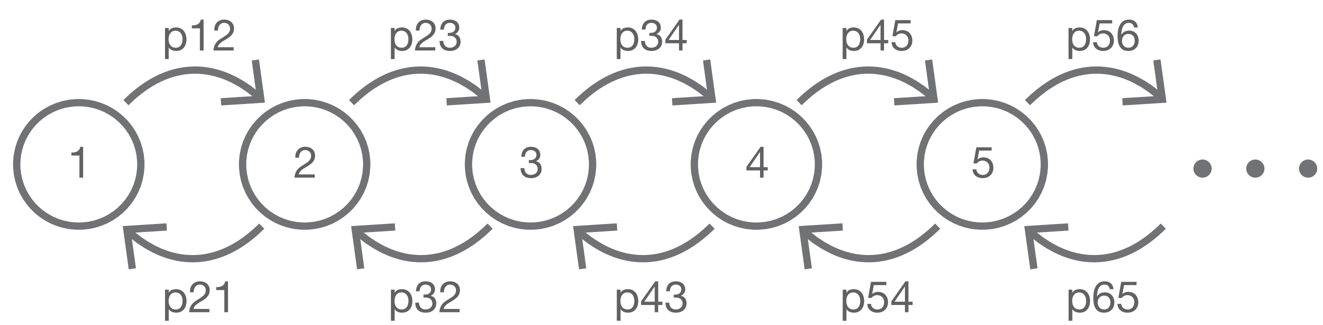
EXAMPLES

Example 2: (Mutating virus)
A virus can exist in N different strains. At each generation the virus mutates with probability $\alpha \in (0, 1)$ to another strain which is chosen at random. Very (medically) relevant question: what is the probability that the strain in the n-th generation of the virus is the same as that in the 0-th?



EXAMPLES

Example 3: (Population dynamics)
Imagine a population with two possible actions. ‘Birth’ an additional individual is added. ‘Death’ an individual is removed. What is the probability that we have N individuals at time t?



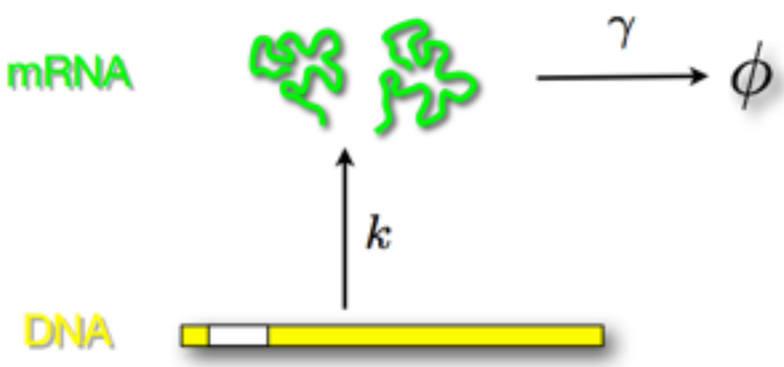
EXAMPLES

Example 4: (Google ranking)
The Google ranking algorithm is based on a Markov chain random surfing of the web.

When visiting any one page, pick the next page either at random with probability q, or at random from the away links with probability 1-q.

Calculate the fraction of times a page will be visited as the number of total visits goes to infinite. This is the page ranking.

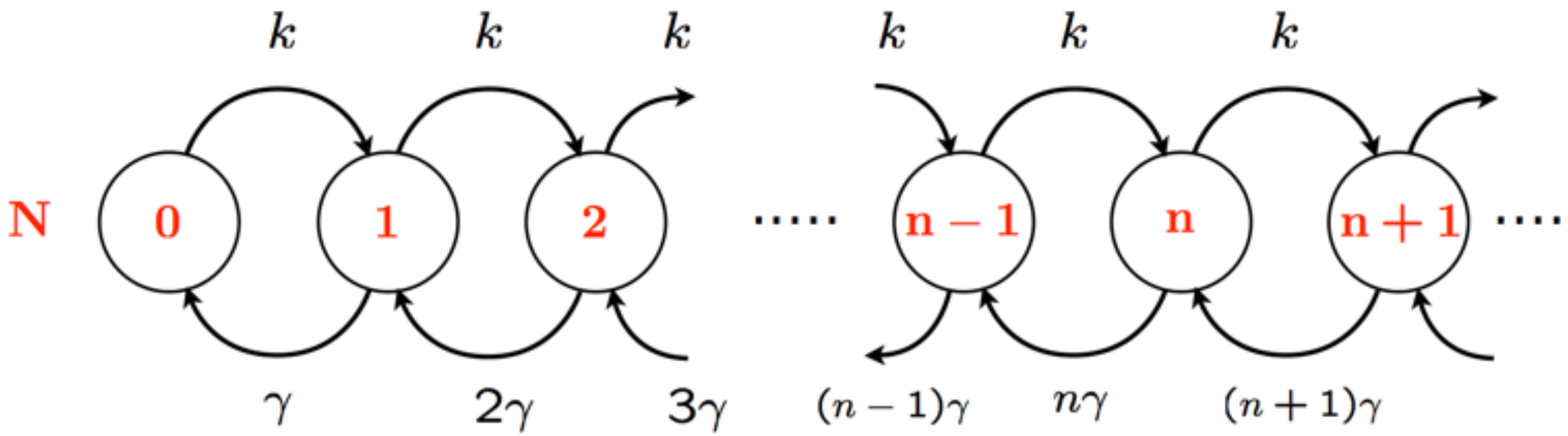
Example 4: (Gene expression)



mRNA copy number $N(t)$ is a random variable

Transcription: Probability a single mRNA is transcribed in time dt is $k\ dt$

Degradation: Probability a single mRNA is degraded in time dt is $n\gamma dt$



EXAMPLES

Definition: (Markov Chain)

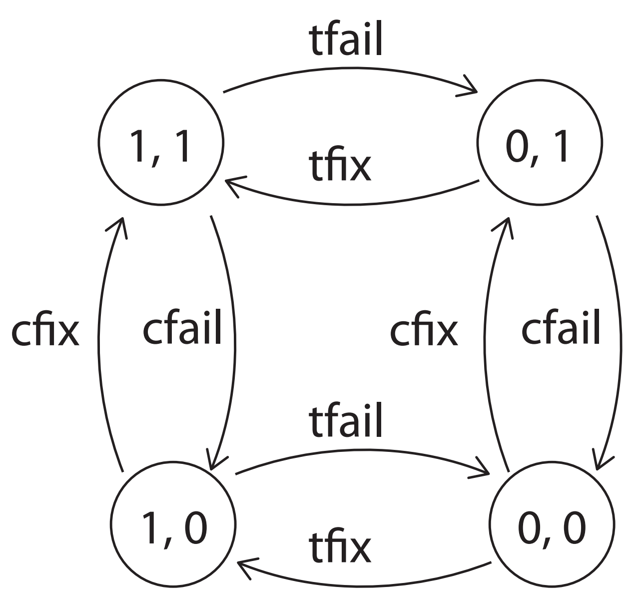
State after n events: $q_n \in \{1, 2, ..., K\}$
Initial state: $q_0 = i$, with probability $P(q_0 = i)$

Markov property/assumption:

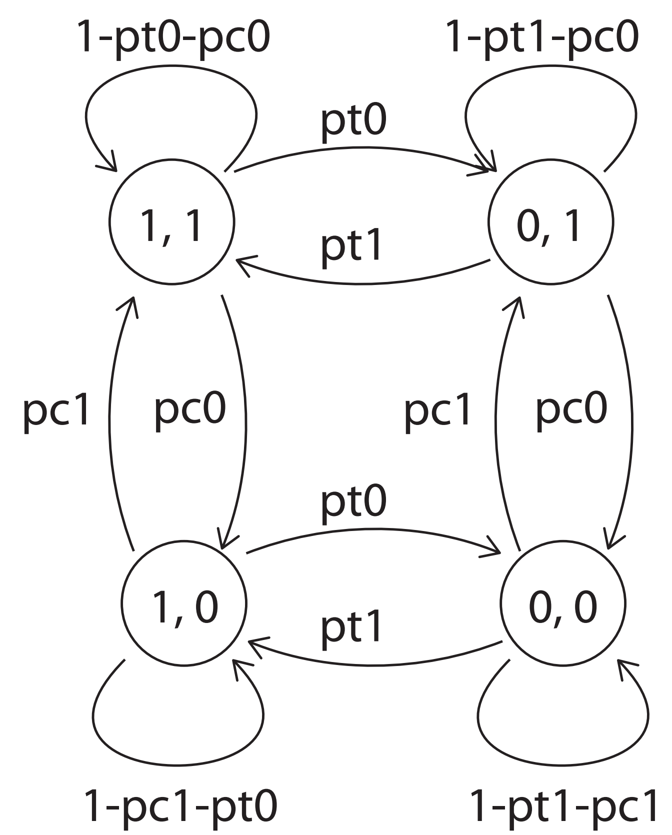
$$\begin{aligned} p_{ij} &= P\left(q_{n+1} = j | q_n = i\right) \\ &= P\left(q_{n+1} = j | q_n = i, q_{n-1}, q_{n-2}, ..., q_0\right) \end{aligned}$$

MARKOV CHAIN ASSUMPTION

Tap and Capacitor fail model
(ignoring all deterministic events)



Corresponding stochastic process model.

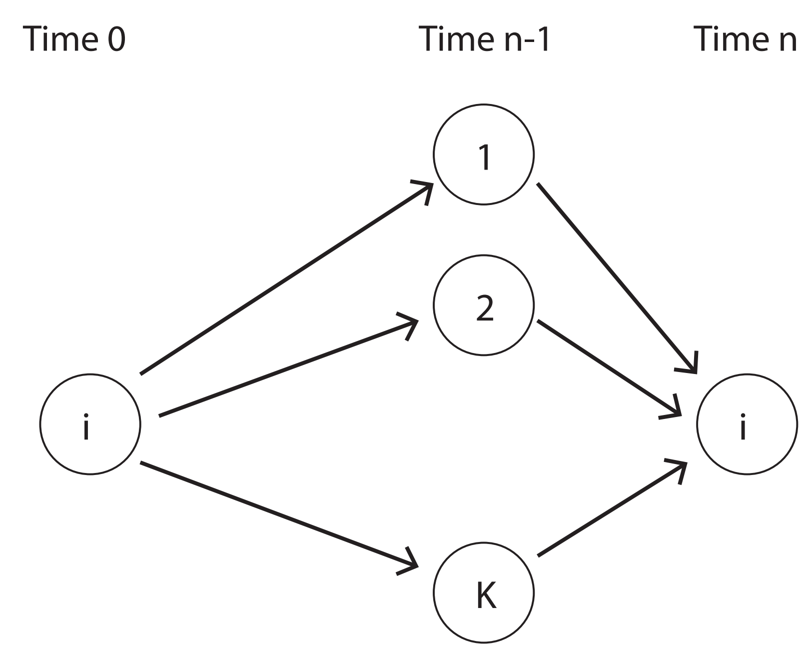


CASE STUDY EXAMPLE

Definition: (n-step transition probabilities)

State occupancy probabilities,
given initial state i:

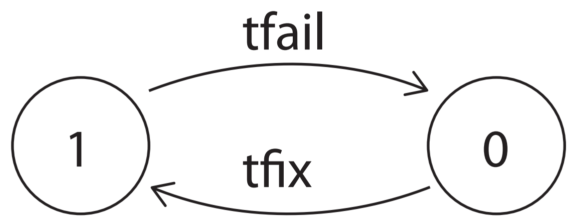
$$r_{ij}(n) = P(q_n = j | q_0 = i)$$



Recurrent formula: $r_{ij}(n) = \sum_{k=1}^{K} r_{ik}(n-1)\, p_{kj}$
Probabilities at time n: $P(q_n = j) = \sum_{k=1}^{K} r_{kj}(n)\, P(q_0 = k)$

TRANSITION PROBABILITIES

Tap fail model
(ignoring all deterministic events)



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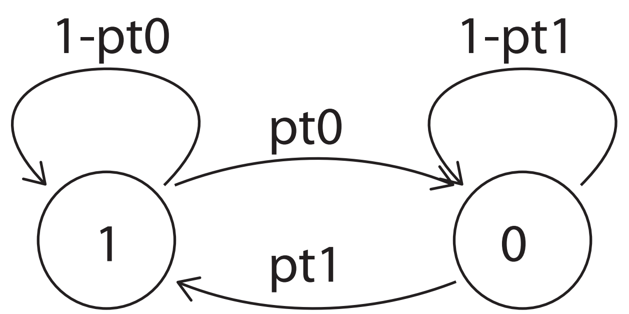


pt0 = .2; pt1 = .3;

	n = 1	n = 2	n = 100	n = 101
r11(n)				
r12(n)				
r21(n)				
r22(n)				

CASE STUDY EXAMPLE

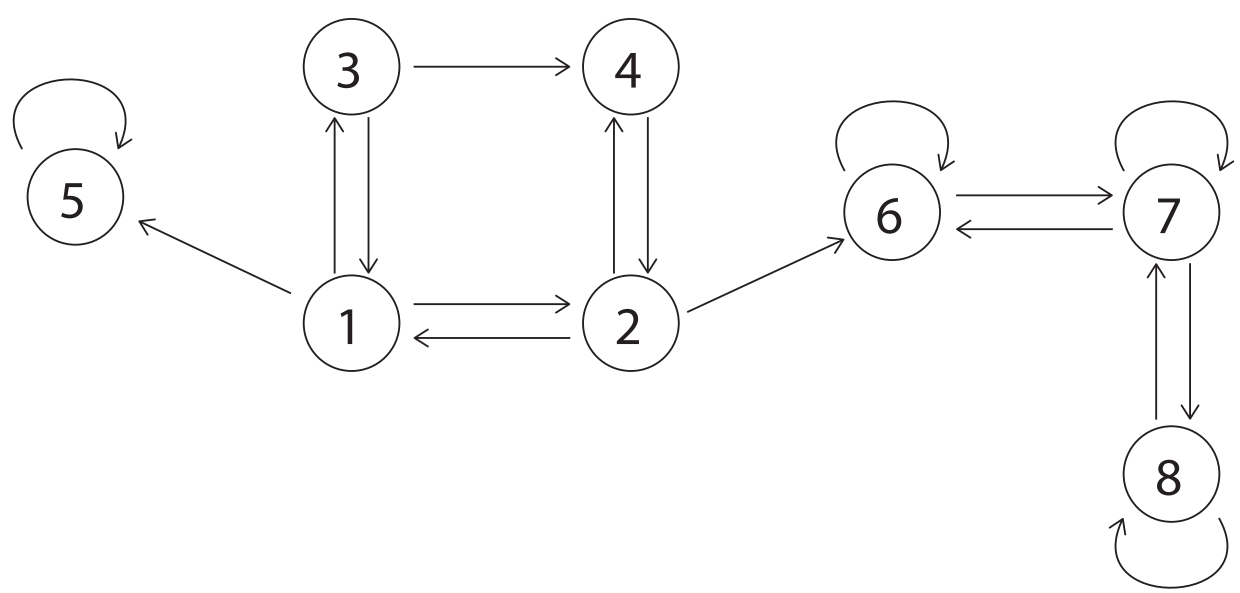
Problem: (convergence)
Does rij(n) always converge to something?

Problem: (initial state)
Does the limit ever depend on the the initial state?

CONVERGENCE QUESTIONS

Definition: (recurrent state)
State i is recurrent if:
starting from i, and from wherever you can go, there is a way of returning to i

Definition: (transient state)
If a state is not recurrent, it is called transient



RECURRENCE AND TRANSIENCE

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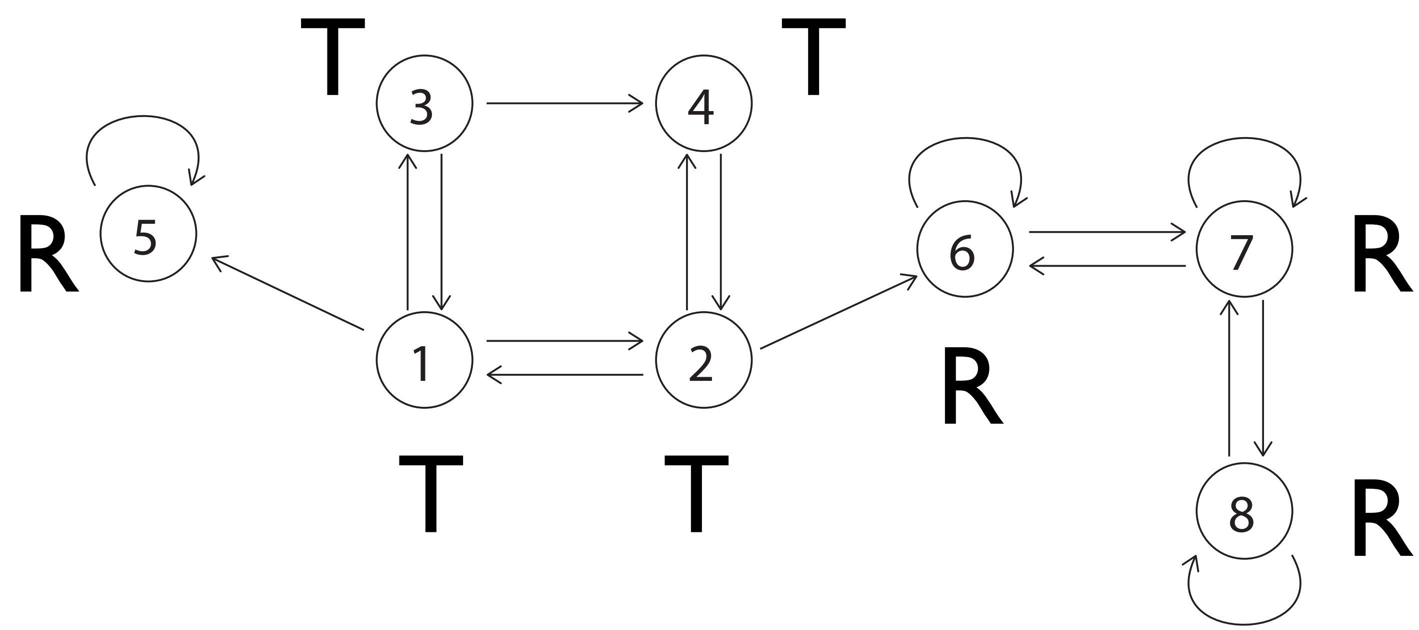


RECURRENCE AND TRANSIENCE

Definition: (recurrent state) $P(q_n = i) \rightarrow 0$

State i is recurrent if:

starting from i, and from wherever you can go, there is a way of returning to i

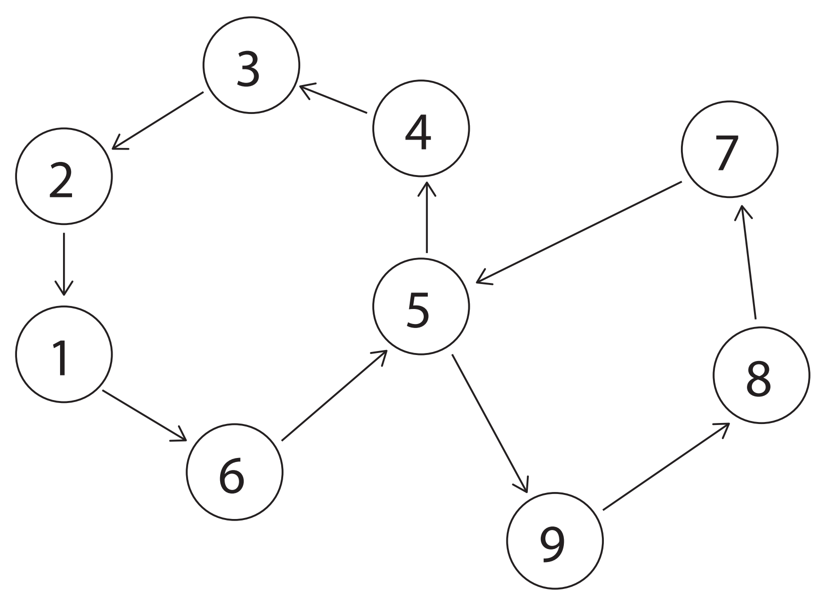
Definition: (transient state)

If a state is not recurrent, it is called transient. A recurrent class is a collection of states that "communicate" with each other.

RECURRENCE AND TRANSIENCE

Definition: (periodic)
Markov chain is called periodic if states can be lumped into clusters so that transitions from one cluster always lead to another cluster.

Questions:
Is the following Markov Chain periodic?



PERIODICITY

Definition: (ergodic)
An aperiodic Markov chain where all states are recurrent is called ergodic.

Theorem: (stationary distribution)
An ergodic Markov chain has a unique distribution.

ERGODICITY

$$\begin{pmatrix} \pi_1(k+1) \\ \pi_2(k+1) \\ \vdots \\ \pi_n(k+1) \end{pmatrix} = \begin{pmatrix} p11 & p12 & \cdots & p1n \\ p21 & \ddots & & \vdots \\ \vdots & & & \\ pn1 & \cdots & & pnn \end{pmatrix} \begin{pmatrix} \pi_1(k) \\ \pi_2(k) \\ \vdots \\ \pi_n(k) \end{pmatrix}, \pi(0) = \pi_0$$

Prob. k+1 | Transition matrix | Prob. k | Initial prob.

$$\pi(k+1) = P\pi(k)$$

TRANSITION MATRIX NOTATION

$$\begin{pmatrix}\pi_1(k+1)\\ \pi_2(k+1)\\ \vdots\\ \pi_n(k+1)\end{pmatrix} = \begin{pmatrix} p11 & p12 & \cdots & p1n\\ p21 & \ddots & & \vdots\\ \vdots & & & \\ pn1 & \cdots & & pnn\end{pmatrix}\begin{pmatrix}\pi_1(k)\\ \pi_2(k)\\ \vdots\\ \pi_n(k)\end{pmatrix}, \pi(0) = \pi_0$$

Prob. k+1 | Transition matrix | Prob. k | Initial prob.

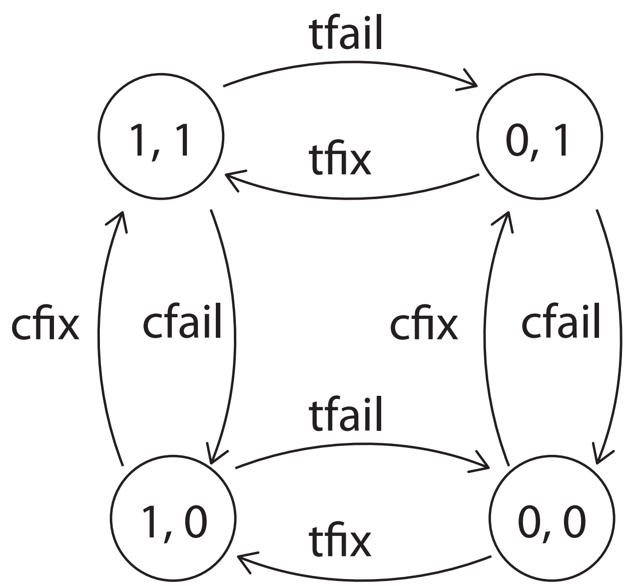
$$\pi(k+1) = P\pi(k)$$

Fact: (n-transition probabilities)
Transition probability $r_{i,j}(n) = (P^n)_{j,i}$

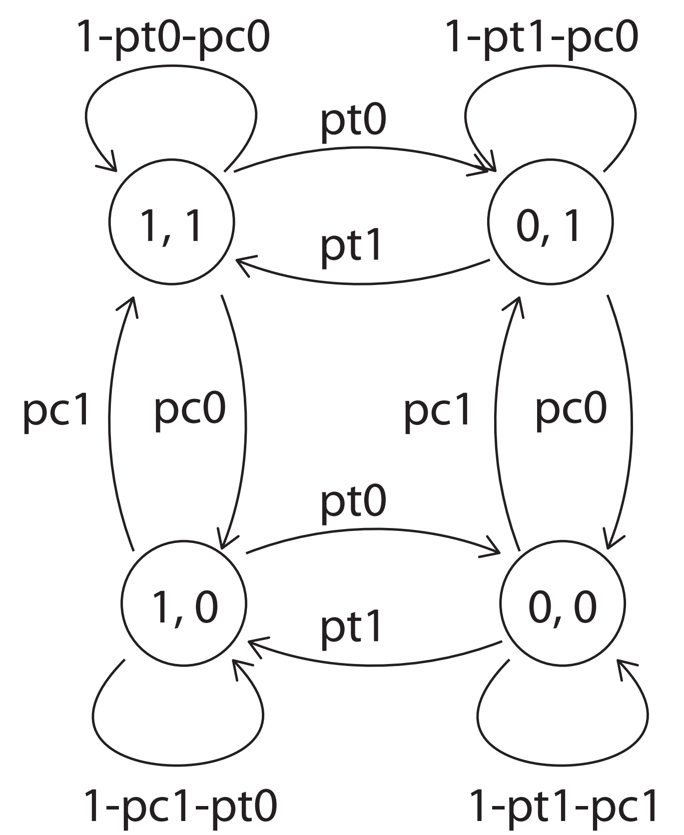
Fact: (stationary distribution)
A stationary distribution satisfies $\pi = P\pi$
For an ergodic Markov Chain there is a unique vector that sums to one and satisfies this equation.

TRANSITION MATRIX NOTATION

Tap and Capacitor fail model
(ignoring all deterministic events)



Corresponding stochastic process model.



CASE STUDY EXAMPLE

Derivation: (Markov Reaction Time)

Probability an event will occur in time $[t, t+\Delta t) : w\Delta t + \mathcal{O}\left(\Delta t\right)^2$
Probability an event will not occur in time $[t, t+\Delta t) : 1 - w\Delta t + \mathcal{O}\left(\Delta t\right)^2$
Probability an event will not occur in time $[t, t+2\Delta t) : \left(1 - w\Delta t + \mathcal{O}\left(\Delta t\right)^2\right)^2$
Define $\tau = K\Delta t$, the the probability an event will not occur in time $[t, t+\tau) : \left(1 - w\frac{\tau}{K} + \mathcal{O}\left(K^{-2}\right)\right)^K$

Taking the limit as $\Delta t \to 0$, probability no event will occur in time $[t, t+\tau) : \lim_{K\to\infty}\left(1 - w\frac{\tau}{K} + \mathcal{O}\left(K^{-2}\right)\right)^K = exp\left(-w\tau\right)$

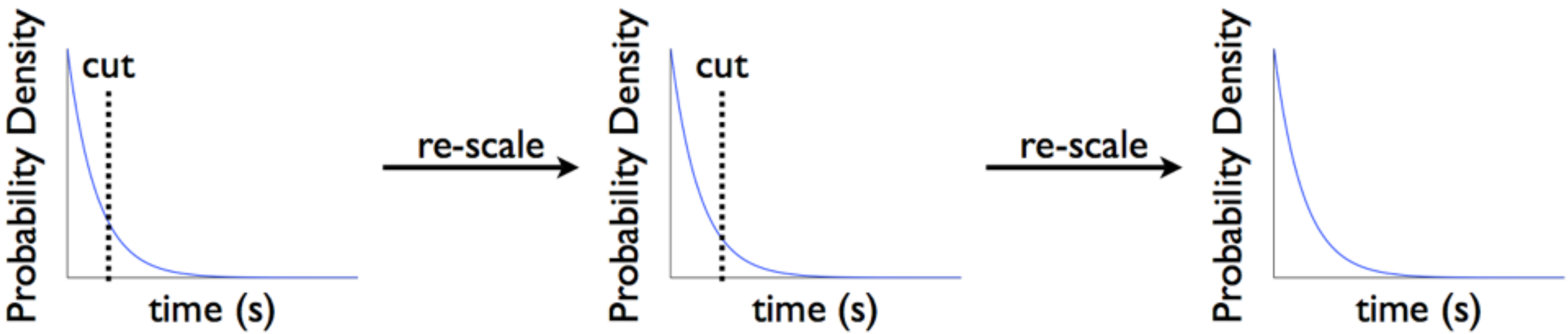
The probability an event will occur in the interval

$$[t, t+\tau) : F_T\left(\tau\right) = 1 - exp\left(-w\tau\right)$$

This is a cumulative distribution of an exponentially distributed random variable T with the density $f_T\left(\tau\right) = \frac{1}{w}exp\left(-w\tau\right)$

CONTINUOUS TIME MC

Derivation: (Memoryless even time)
Exponential waiting times are memoryless.



EXPONENTIAL EVENT TIMES

Derivation: (Continuous time transition probabilities)

Transition probabilities satisfy $\frac{dR(t)}{dt} = W \ R(t)$

with $W = \begin{pmatrix} w_{1,1} & w_{2,1} & \cdots \\ w_{1,2} & \ddots & \\ \vdots & & \end{pmatrix}$, where for $i \neq j, w_{i,j}$ is the

propensity of going to state *j* from state *i*. The diagonal elements of *Q*, $w_{i,i} = -\sum_{j \neq i} w_{i,j}$.

The matrix $R(t) = \begin{pmatrix} r_{1,1}(t) & r_{2,1}(t) & \cdots \\ r_{1,2}(t) & \ddots & \\ \vdots & & \end{pmatrix}$, $r_{i,j}(t) = P(q(t) = j | q(0) = i)$

satisfies $R(t) = e^{tQ}$

TRANSITION PROBABILITIES