MARKOV CHAINS: first step conditioning and detailed balance

1. **n-step transition matrix**

   Consider transition matrix $P = (p_{ij})$, where $p_{ij} = P(q_{n+1} = j|q_n = i)$. Then the n-step transition probability $r_{ij}(n) = P(q_n = j|q_0 = i)$ can be computed from the n-step transition matrix P^n. Simply, $r_{ij}(n) = (P^n)_{ij}$, ie the ith column and the jth row of P^n.

2. **first step analysis: hitting time**

   Consider the set of states Q, then for $A \subset Q$, the hitting time

   $$T_A = \min\{t \geq 0|q_t \in A\} \tag{1}$$

   can be studied through its expectation function as follows.

   Let

   $$h(i) = \mathbb{E}(T_A|q_0 = i) = \mathbb{E}_i T_A. \tag{2}$$

   By conditioning on the first step,

   $$h(i) = 0, i \in A$$
   $$h(i) = 1 + \sum_j p_{ij} h(j).$$

3. **first step analysis: hitting time comparison**

   Consider disjoint subsets $A, B \subset Q$ and the comparison function $g(i) = P(T_A < T_B|q_0 = i)$. We have

   $$g(i) = 1, i \in A; g(i) = 0, i \in B \tag{3}$$

   and by conditioning on the first step

   $$g(i) = \sum_j p_{ij} g(j), i \in Q \setminus (A \cup B) \tag{4}$$

4. **Example: consecutive throws of a die** Throw a fair die until getting two 6's in succession. This requires some random number of T throws. Calculate $\mathbb{E}T$.

   This can be done by defining a Markov chain as follows.

   $q_n = 0$ if the tth throw is not 6.

   $q_n = 1$ if the tth throw is a 6 but the previous throw was not a 6.

   $q_n = 2$ if the $t$the throw was a 6 and the previous throw was also a 6.

   This chain has states $\{0, 1, 2\}$ and

   $$p_{00} = 5/6, p_{01} = 1/6, p_{10} = 5/6, p_{12} = 1/6. \tag{5}$$

   We can set up the equations for $h(i) = \mathbb{E}_i T_2$. We find

   $$h(1) = 36, h(0) = 42. \tag{6}$$

   Hence the answer is 42.

5. **Example: Gambler's Ruin**

   In words, you start with i dollars and bet 1 dollar each step, with probability p of winning, and ocntinue until reach K, at which point you stop, or until you go bust.

   The states $Q = \{0, 1, ...K\}$ and $p_{i,i+1} = p, p_{i,i-1} = 1 - p, 1 \leq i \leq K - 1$. The states 0 and K are absorbing, so that $p_{00} = 1, p_{KK} = 1$.

   We want to know $g(i) = P(T_K < T_0|q_0 = i)$ and $h(i) = \mathbb{E}_i T_{\{0,K\}}$.

   Solve this numerically for the fair coin example and the roulette example where you bet on a single color. Use the following initial conditions and absorbing states: $i = 10, K = 20$, $i = 20, K = 40$, $i = 100, K = 100$.

6. **finding stationary distributions: 3x3 example**

Suppose the transition probability matrix is given by

$$P = \begin{pmatrix} 0.7 & 0.4 & 0 \\ 0.2 & 0.6 & 1 \\ 0.1 & 0 & 0 \end{pmatrix}. \tag{7}$$

It can be shown that the corresponding Markov chain is ergodic. The stationary distribution therefore exists and is unique.

We can find the distribution by solving the equation $\pi = P\pi$. In Matlab, use the command

$$[V, D] = eig(P). \tag{8}$$

Find the diagonal element of D, such that $d_{ii} = 1$ (such element is guaranteed to exist by ergodicity). Next take v_i, the ith column of V, and compute the stationary distribution $\pi = v_i / \sum v_i$.